S_3 - MAT 203 (CS, IT)

DISCRETE MATHEMATICAL STRUCTURES (DMS)

Text Book: Discrete & Combinatorial Mathematics by Ralph P Grimaldi ; B.V Ramana.

MODULE : 1 (9 hours)

FUNDAMENTALS OF LOGIC

Syllabus:

* Mathematical Logic
* Basic Connectives & truth tables.
* Statements
* Logical Connectives.
* Tautology, Contradiction.
* Logical Equivalence.
* The Laws of logic.
* The principle of duality.
* Substitution Rules.
* The implication - The contrapositive
* The converse, The inverse.
* Logical implication
* Rules of inference
* The use of Quantifiers
* Open Statement, Quantifier,
* Logically equivalent
* Contra positive, Converse, inverse
* Logical equivalence & implication of quantified
* statement, Implication, Negation

Logic deals with the method of reasoning. One of the aims of logic is to provide rules by which we can determine whether a particular reasoning or argument is valid. It was the English Mathematician George Boole, who introduced this.

Propositions (Statements)

A statement or proposition is a declarative sentence which is either true or false, but not both.

'True' and 'False' are called truth values and are denoted by (1 or T) and (0 or F) respectively.

We use lowercase letters of the alphabet to represent statements.

eg:- p, q and r. etc.

Sentences which are exclamatory, introgative or imperative are not propositions, since they do not have truth values.

eg:- p: New Delhi is the capital city of India - 1

q: $2+2=3$ - 0

r: $x+y+3=z$ not a proposition.

s: $9<6$ - 0

t: What a beautiful place! → exclamatory not a proposition

u: What is the time now? → introgative "

v: Get up and do your exercises. → command "

w: Please give me your pen. → imperative "

Primitive Satements & Compound Statements

A proposition is said to be primitive if it cannot be broken down into simpler propositions. ~~Riii~~ They are state ments with out connectives.

eg:- Vazhakulam is in EKM district

$2+1=3$

A compound statement is the combination of 2 or more primitive statements. They are statements with connectives.

eg:- Roses are red and violets are blue.

Logical Connectives.

The words or phrases used to connect two or more primitive statements are called connectives

eg:- or, and, if... then, if and only if

Truth Table

A table giving truth values of a compound statement in terms of its component part is called the truth table.

Note:- If there are 'n' distinct components in a statement formula, we have 2^n possible combinations of the truth values in the truth table.

Q) How many rows are needed for the truth table of the compound statement $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t]$, where p, q, r, s, t are primitive statements.

Ans:-

How to construct truth tables

Case ① – with 1 primitive statement (2^1 rows) (2 rows)

P
1
0

Case ② with 2 primitive statements (2^2 rows) 4 rows

P	q
1	1
1	0
0	1
0	0

Case ③ with 3 primitive statements (2^3 rows) (8 rows)

P	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

or we can write 4 zeros first, then 4 ones in 1st column
In the second column, 2 zeros 2 ones respectively
Last column 0, 1, 0, 1, alternatively

Case ④ p, q, r, s 2^4 rows (16 rows).

LOGICAL CONNECTIVES

Symbol	Name	Connective word or phrase
1) $\neg$, $\sim$	Negation ($\neg p$)	not
2) $\wedge$	Conjunction ($p \wedge q$)	and
3) $\vee$	Disjunction (inclusive) ($p \vee q$)	or
4) $\underline{\vee}$	Exclusive Disjunction ($p \underline{\vee} q$)	or
5) $\rightarrow$	Implication ($p \rightarrow q$)	* if p then q, implies * p only if q, * p is sufficient for q * q is necessary for p.
6) $\leftrightarrow$	Biconditional	* p if and only if q (or p iff q) * p is necessary & sufficient for q

Truth Tables

1) Negation

p	¬p
1	0
0	1

2) Conjunction (and) $p \wedge q$

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

3) Disjunction (or) $p \vee q$

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

4) Implication (if... then) $(p \rightarrow q)$

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

5) Biconditional (if and only if) $(p \leftrightarrow q)$

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Truth Tables (All together)

1) Negation

p	$\neg p$
0	1
1	0

The column of truth values of p and $\neg p$ are oposite to each other

		Conjunction	Disjunction	Exclusive disjunction	Implication	Biconditional
p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Axiom of conjunction $(p \wedge q)$

$p \wedge q$ is true only when both p & q are true

Axiom of Disjunction $(p \vee q)$

$p \vee q$ is false only when both p & q are false.

Axiom of exclusive disjunction $(p \veebar q)$

$p \veebar q$ is true only when exactly one of p, q is true.

Axiom of Implication $(p \rightarrow q)$

$p \rightarrow q$ is false when p is true and q is false.
ie A true statement doesnot imply a false statement.

Axiom of Biconditional $p \leftrightarrow q$

$p \leftrightarrow q$ is true when both p & q have same truth values.

EXERCISES 2.1

1. Determine whether each of the following sentences is a statement.

   a) In 2003 George W. Bush was the president of the United States.

   b) $x + 3$ is a positive integer.

   c) Fifteen is an even number.

   d) If Jennifer is late for the party, then her cousin Zachary will be quite angry.

   e) What time is it?

   f) As of June 30, 2003, Christine Marie Evert had won the French Open a record seven times.

2. Identify the primitive statements in Exercise 1.

3. Let p, q be primitive statements for which the implication $p \rightarrow q$ is false. Determine the truth values for each of the following.

   a) $p \wedge q$ b) $\neg p \vee q$ c) $q \rightarrow p$ d) $\neg q \rightarrow \neg p$

4. Let p, q, r, s denote the following statements:

   p: I finish writing my computer program before lunch.
   q: I shall play tennis in the afternoon.
   r: The sun is shining.
   s: The humidity is low.

   Write the following in symbolic form.

   a) If the sun is shining, I shall play tennis this afternoon.

   b) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon.

   c) Low humidity and sunshine are sufficient for me to play tennis this afternoon.

5. Let p, q, r denote the following statements about a particular triangle ABC.

   p: Triangle ABC is isosceles.
   q: Triangle ABC is equilateral.
   r: Triangle ABC is equiangular.

   Translate each of the following into an English sentence.

   a) $q \rightarrow p$ b) $\neg p \rightarrow \neg q$

   c) $q \leftrightarrow r$ d) $p \wedge \neg q$

   e) $r \rightarrow p$

Q) Construct the truth table for

1) $q \wedge (\neg r \rightarrow p)$

2) $p \vee (q \wedge r)$

3) $(p \vee q) \wedge r$

p	q	r	$\neg r$	① $\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$ $q \wedge ①$	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	1	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0	1	0
0	1	1	0	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1	0
1	0	1	0	1	0	0	1	1	1
1	1	0	1	1	1	0	1	1	0
1	1	1	0	1	1	1	1	1	1

Tautology and Contradiction.

A compound statement that is always true for all possible truth values of its variables or in other words contain only 1 in the last column of its truth table is called a tautology, denoted by the symbol T_0

A compound proposition that is always false for all possible values of its variables or in other words, contain only 0 in the last column of its truth table is called a contradiction, denoted by the symbol, F_0

Q Show that the implication $\neg(p \to q) \to \neg q$ is a tautology.

Ans

p	q	$\neg q$	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to \neg q$
1	1	0	1	0	1
1	0	1	0	1	1
0	1	0	1	0	1
0	0	1	1	0	1

The truth values of the last column are all 1's (true). Hence it is a tautology (T_0)

Q. Show that $p \wedge \neg p$ is a contradiction.

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

The truth values of the last column are all 0's (false). Hence it is a contradiction (F_0)

Note: 1 A proposition that is neither a tautology nor a contradiction is called a contingency.

Note: 2 The negation of a tautology is a contradiction & the negation of a contradiction is a tautology.

H.W Construct the truth table for the following which of them are tautologies and contradictions.

1) $(p \to q) \wedge [(q \wedge \neg r) \to (p \vee r)]$

2) $p \wedge (\neg p \wedge q)$

3) $p \to (p \vee q)$

4) $\neg (p \vee \neg q) \to \neg p$

5) $q \leftrightarrow (\neg p \vee \neg q)$

6) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

7) $(p \to q) \to r$

H.W Q. Verify that $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology.

Q) If statement q has the truth value 1, determine all truth value assignments for the primitive statements p, r and s for which the truth value of the statement

$$[q \to [(\neg p \vee r) \wedge \neg s]] \wedge [\neg s \to (\neg r \wedge q)]$$

4-variables, $p\ q\ r\ s$

no of rows $= 2^4 = \underline{\underline{16}}$

p	q	r	s	$\neg p$	$\neg r$	$\neg s$	① $\neg p \vee r$	③ ①$\wedge \neg s$	② $\neg r \wedge q$	④ $\neg s \to$②	⑤ $q \to$③	④$\wedge$⑤
1	1	1	1	0	0	0	1	0	0	1	0	0
1	1	1	0	0	0	1	1	1	0	0	1	0
1	1	0	1	0	1	0	0	0	1	1	0	0
1	1	0	0	0	1	1	0	0	1	1	0	0
1	0	1	1	0	0	0	1	0	0	1	1	1
1	0	1	0	0	0	1	1	1	0	0	1	0
1	0	0	1	0	1	0	0	0	0	1	1	1
1	0	0	0	0	1	1	0	0	0	0	1	0
0	1	1	1	1	0	0	1	0	0	1	0	0
0	1	1	0	1	0	1	1	1	0	0	1	0
0	1	0	1	1	1	0	1	0	1	1	0	0
0	1	0	0	1	1	1	1	1	1	1	1	1 ✓
0	0	1	1	1	0	0	1	0	0	1	1	1
0	0	1	0	1	0	1	1	1	0	0	1	0
0	0	0	1	1	1	0	1	0	0	1	1	1
0	0	0	0	1	1	1	1	1	0	0	1	0

Truth value of q is 1 (given).

The given compound statement is 1 when $p:0$, $r:0$ and $s:0$

H.W) Construct the truth table for

$$[p \to (q \to s)] \wedge (\neg r \vee p) \wedge q$$

Exercise 2.1 (Answers) (page no: 911 in Text book)

1) (a), (c), (d) (f) are statements.

2) (a), (c) (f),

3)

p	q	$p \to q$	$p \wedge q$	$\neg p \vee q$	$q \to p$	$\neg q \to \neg p$
1	1					
1	0	0	0	0	1	0
0	1					
0	0					

given

4) a) $r \to q$
b) $q \to r$
c) $s \to q$

5) $q \to p$: If the Δle ABC is equilateral then Δle ABC is isosceles

$\neg p \to \neg q$: If the Δle ABC is not isosceles then it is not equilateral.

$q \leftrightarrow r$: Δle ABC is equilateral iff it is equiangular.

$p \wedge \neg q$: Δle ABC is isosceles and it is not equilateral.

$r \to p$: If Δle ABC is equiangular then it is isosceles.

3, 4, 6 Tautologies 1, 5, 7 contingency
2 Contradiction

Answers (page no 9)

1)

p	q	r	① p→q	¬r	② q∧¬r	③ pVr	④ ②→③	①∧④
1	1	1	1	0	0	1	1	1
1	1	0	1	1	1	~~0~~ 1	~~0~~ 1	~~0~~ 1
1	0	1	0	0	0	1	1	0
1	0	0	0	1	0	~~0~~ 1	1	0
0	1	1	1	0	0	~~0~~ 1	1	1
0	1	0	1	1	1	0	0	0
0	0	1	1	0	0	1	1	1
0	0	0	1	1	0	0	1	1

Qn: 1

2)
3)

p	q	¬p	¬p∧q	p∧(¬p∧q)	pVq	p→pVq
1	1	0	0	0	1	1
1	0	0	0	0	1	1
0	0	1	0	0	0	1
0	1	1	1	0	1	1

Qn: 2 Qn: 3

4)
5)

p	q	¬p	¬q	pV¬q	① ¬(pV¬q)	①→¬p	② ¬pV¬q	q↔②
1	~~0~~ 1	0	0	1	0	1	0	0
1	0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1	1
0	0	1	1	1	0	1	1	0

Qn: 4 Qn: 5

6)
7)

p	q	r	① p→q	② q→r	③ ①∧②	④ p→r	③→④	(p→q)→r
1	1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1	0
1	0	1	0	1	0	1	1	1
1	0	0	0	1	0	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	1	0	0	1	1	0
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	0

Qn: 6 Qn: 7

5. Construct the truth table for

$$(p \to (q \to s)) \wedge (\neg r \vee p) \wedge q$$

p	q	r	s	$q \to s$	$\overset{a}{p \to (q \to s)}$	$\overset{b}{\neg r \vee p}$	$a \wedge b \wedge q$
T	T	T	T	T	T	T	T
T	T	T	F	F	F	T	F
T	T	F	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	F
T	F	T	F	T	T	T	F
T	F	F	T	T	T	T	F
T	F	F	F	T	T	T	F
F	T	T	T	T	T	F	F
F	T	T	F	F	T	F	F
F	T	F	T	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	F	F
F	F	T	F	T	T	F	F
F	F	F	T	T	T	T	F
F	F	F	F	T	T	T	F

LOGICAL EQUIVALENCE : THE LAWS OF LOGIC

Algebra of propositions.

In this algebra, we shall use the truth tables of the statements to show that two statements are logically equivalent

Defn Two statements S_1, S_2 are said to be logically equivalent iff both have same truth values for all choices of truth values for their primitive components. It is denoted by $\boxed{S_1 \Leftrightarrow S_2}$

Eg:- S.T 1) $P \to q \Leftrightarrow \neg P \vee q$

2) $P \leftrightarrow q \Leftrightarrow (P \to q) \wedge (q \to P)$

P	q	$\neg P$	$\neg P \vee q$	$P \to q$	$q \to P$	$P \leftrightarrow q$	$(P\to q) \wedge q \to P$
1	1	0	1	1	1	1	1
1	0	0	0	0	1	0	0
0	1	1	1	1	0	0	0
0	0	1	1	1	1	1	1

Same truth Values

1) $P \to q \Leftrightarrow \neg P \vee q$

Same truth values

$P \leftrightarrow q \Leftrightarrow (P \to q) \wedge (q \to P)$

Q) Show that $P \wedge (\neg P \vee r)$ and $P \vee (q \wedge \neg r)$ are not logically equivalent.

Hint:- Draw truth table & show that both have different truth values.

The laws of Logic

1) $\neg\neg p \Leftrightarrow p$ Law of Double Negation

2) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ } DeMorgan's Law.

3) $p \vee q \Leftrightarrow q \vee p$
$p \wedge q \Leftrightarrow q \wedge p$ } Commutative Law.

4. $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ } Associative Law.

5. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ } Distributive Law

6. $p \vee p \Leftrightarrow p$
$p \wedge p \Leftrightarrow p$ } Idempotent Laws

7) $p \vee F_0 \Leftrightarrow p$
$p \wedge T_0 \Leftrightarrow p$ } Identity laws.

8) $p \vee \neg p \Leftrightarrow T_0$
$p \wedge \neg p \Leftrightarrow F_0$ } Inverse laws.

9) $p \vee T_0 \Leftrightarrow T_0$
$p \wedge F_0 \Leftrightarrow F_0$ } Domination Laws.

10) $p \vee (p \wedge q) \Leftrightarrow p$
$p \wedge (p \vee q) \Leftrightarrow p$ } Absorption laws.

Note: These laws can easily established using truth tables.

H.W **Exercises 2.2**

3) Let p, q, r denote primitive statements use truth tables to verify the following logical equivalences.

(1) $p \rightarrow (q \wedge r) \Leftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$

(2) $p \rightarrow (q \vee r) \Leftrightarrow [\neg r \rightarrow (p \rightarrow q)]$

(3) $(p \vee q) \rightarrow r \Leftrightarrow (p \rightarrow r) \wedge (q \rightarrow r)$

4) Verify the first absorption law by
H.W means of a truth table.

5)

Dual (S^d)

Let 'S' be a statement contains only $\wedge$ and $\vee$, then the dual of S is denoted by S^d which is obtained from S by replacing each $\wedge$ by $\vee$, each $\vee$ by $\wedge$ each T_0 by F_0 and F_0 by T_0

Eg:- Write the dual of

$$S: (p \wedge \neg q) \vee (r \wedge T_0)$$

dual S^d: $(p \vee \neg q) \wedge (r \vee F_0)$

Note:- $\neg q$ is unaltered as we write S^d

The principle of duality

Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$ if $\boxed{s \Leftrightarrow t \text{ then } s^d \Leftrightarrow t^d}$

1) Q. Write the dual of the logical equivalence in $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$

Ans:- By the principal of duality

$$(\neg p \wedge q) \vee (p \vee (p \vee q)) \Leftrightarrow p \vee q$$

2) Write the dual for

a) $q \rightarrow p$

b) $p \rightarrow (q \wedge r)$

c) $p \leftrightarrow q$

d) $p \veebar q$

Ans a) $q \rightarrow p \Leftrightarrow \neg q \vee p$

$\therefore$ dual of $q \rightarrow p$ is $\underline{\underline{\neg q \wedge p}}$

b)

c) $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$

$\therefore$ Dual of $p \leftrightarrow q$ is $\underline{\underline{(\neg p \wedge q) \vee (\neg q \wedge p)}}$

d) $p \veebar q$

It's dual is $\underline{\underline{p \veebar q}}$

Converse, Inverse & Contrapositive of Implication.

If $p \to q$ then the statement $\boxed{q \to p \text{ is called Converse}}$

If $p \to q$ then $\boxed{\neg p \to \neg q \text{ is called the inverse}}$

If $p \to q$ then $\boxed{\neg q \to \neg p \text{ is called the contrapositive}}$

converse = inverse

p	q	$p \to q$ (implication)	$q \to p$ (converse)	$\neg p$	$\neg q$	$\neg p \to \neg q$ (inverse)	$\neg q \to \neg p$ (contrapositive)
1	1	1	1	0	0	1	1
1	0	0	1	0	1	1	0
0	1	1	0	1	0	0	1
0	0	1	1	1	1	1	1

implication = Contrapositive.

From the truth table

$$p \to q \Longleftrightarrow \neg q \to \neg p$$

$$q \to p \Longleftrightarrow \neg p \to \neg q$$

1) Write the converse, inverse and contrapositive of each of the following implications. For each implication, determine its truth value as well as the truth value of its corresponding converse, inverse & Contrapositive.

a) If $0+0=0$ then $1+1=1$

b) If $-1<3$ and $3+7=10$, then $\sin\left(\frac{3\pi}{2}\right)=-1$

H.W

Ans:- a) p: $0+0=0$ true
q: $1+1=1$ false

$p \to q$ means $1 \to 0$, truth value is FALSE

Converse is $q \to p$
ie $0 \to 1$, ∴ truth value is TRUE

Inverse is $\neg p \to \neg q$
$0 \to 1$, truth value is TRUE

Contrapositive $\neg q \to \neg p$
$1 \to 0$, truth value is FALSE

Negation of a compound statement

1) Negate the following and express them in smooth English.

a) Norma is doing her homework, and Karen is practicing her piano lessons.

Ans:- p: Norma is doing her homework
q: Karen is practicing her piano lessons.

The given statement is $p \wedge q$

Negation of the statement is $\neg(p \wedge q)$
$\Longleftrightarrow \neg p \vee \neg q$

ie Norma is not doing her homework or Karen is not practicing her piano lessons.

HW b) Kelsey will get a good education, if she puts her studies before her interest in cheerleading.

Ans:- p: Kelsey puts her studies before her interest in cheerleading

q:- She will get a good education.

Statement is $p \rightarrow q$.

$$p \rightarrow q \Leftrightarrow \neg p \vee q$$

$$\neg(p \rightarrow q) \Leftrightarrow \neg(\neg p \vee q)$$

$$\Leftrightarrow (p \wedge \neg q)$$

ie Kelsey put her studies before her interest in cheerleading, but (and) she did not get a good education

c) If Harold passes his C++ course and
H.W. finishes his data structure project, then he will graduate at the end of the semester

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HW ANSWERS

q	r	$q \wedge r$	$p \to q \wedge r$	① $p \to q$	② $p \to r$	① $\wedge$ ②
T	T	T	T	T	T	T
T	F	F	F	T	F	F
F	T	F	F	F	T	F
F	F	F	F	F	F	F
T	T	T	T	T	T	T
T	F	F	T	T	T	T
F	T	F	T	T	T	T
F	F	F	T	T	T	T

$q \vee r$	$p \to q \vee r$	$\neg r$	$\neg r \to (p \to q)$
T	T	F	T
T	T	T	T
T	T	F	T
F	F	T	F
T	T	F	T
T	T	T	T
T	T	F	T
F	T	T	T

$p \vee q$	$(p \vee q) \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T
T	F	F	F
T	T	T	T
T	F	T	F
T	T	T	T
T	F	F	F
F	T	T	T
F	T	T	T

4) **Absorption laws**

$p \wedge (p \vee q) \Leftrightarrow p$

$p \vee (p \wedge q) \Leftrightarrow p$

p	q	$p \vee q$	$p \wedge q$	$p \wedge (p \vee q)$	$p \vee (p \wedge q)$
T	T	T	T	T	T
T	F	T	F	T	T
F	T	T	F	F	F
F	F	F	F	F	F

page No: 16 Home work answer

1 b)

p: $-1<3$ TRUE
q: $3+7=10$ TRUE
r: $\sin\frac{3\pi}{2}=-1$ TRUE

The given statement is $(p\wedge q)\rightarrow r$

$p\wedge q$ is TRUE, $T\rightarrow T \Leftrightarrow T$
$1\rightarrow 1 = 1$ TRUE

converse is $r\rightarrow(p\wedge q)$
$1\rightarrow 1$ is 1 TRUE

Inverse is $\neg(p\wedge q)\rightarrow \neg r$
$F\rightarrow F$
$0\rightarrow 0$ is TRUE

Contra positive is $\neg r\rightarrow\neg(p\wedge q)$
$0\rightarrow 0$ is TRUE

page No: 18 Home work answer

p: Harold passes his C++ Course
1) c) q: He finishes his data structure project
r: He will graduate at the end sem.

Ans:- The given statement is $(p\wedge q)\rightarrow r$

$$(p\wedge q)\rightarrow r \Leftrightarrow \neg(p\wedge q)\vee r$$

It's negation is

$$\neg[(p\wedge q)\rightarrow r] \Leftrightarrow \neg(\neg(p\wedge q)\vee r)$$
$$\Leftrightarrow (p\wedge q)\wedge\neg r$$

ie Harold passes his C++ Course and finishes his data structure project, but (and) he ~~couldn't~~ didnot graduate at the end semester.

Q) For primitive statements p, q, r, s simplify the compound statements.

$$\Big[\big[[(p\wedge q)\wedge r] \vee [(p\wedge q)\wedge \neg r]\big] \vee \neg q\Big] \rightarrow s$$

$$\Leftrightarrow \Big[\big[(p\wedge q)\wedge \cancel{(r\vee \neg r)}^{T_0}\big] \vee \neg q\Big] \rightarrow s \quad \text{(Distributive law)}$$

$$\Leftrightarrow [(p\wedge q)\vee \neg q] \rightarrow s \qquad \text{(Domination law)}$$

$$\Leftrightarrow \neg[(p\wedge q)\vee \neg q] \vee s \qquad p\rightarrow q \Leftrightarrow \neg p \vee q$$

$$\Leftrightarrow [\neg(p\wedge q)\wedge q] \vee s \qquad \text{(Demorgan's law)}$$

$$\Leftrightarrow [(\neg p\vee \neg q)\wedge q] \vee s \qquad \text{(Distributive law)}$$

$$\Leftrightarrow [(\neg p\wedge q) \vee \cancel{(\neg q\wedge q)}^{F_0}] \vee s \quad \text{(Law of domination)}$$

$$\Leftrightarrow \underline{\underline{(\neg p\wedge q)\vee s}}$$

Q) With out constructing truth table prove that the following is a tautology.

a) $[p\vee(q\wedge r)] \vee \neg[p\vee(q\wedge r)]$

b) $[(p\vee q)\rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p\vee q)]$

Ans:- a) Let $p \vee (q \wedge r)$ be a

$\therefore a \vee \neg a \Leftrightarrow T_0$ By the law of inverse

b) $(p \vee q) \rightarrow r \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$

$\Leftrightarrow \neg(p \vee q) \vee r \leftrightarrow r \vee \neg(p \vee q)$

[Let $\neg(p \vee q) \vee r$ be a]

$\Leftrightarrow a \leftrightarrow a$

By ("iff" truth table)

$\Leftrightarrow T_0$

$p \rightarrow q \Leftrightarrow \neg p \vee q$

Problems

1. Negate each of the following and simplify the resulting statement

a) $p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)$

b) $(p \wedge q) \rightarrow r$

H.W c) $p \vee q \vee (\neg p \wedge \neg q \wedge r)$

Ans:- a) $\neg \left[p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r) \right]$

$\Longleftrightarrow \neg p \vee \neg (q \vee r) \vee \neg (\neg p \vee \neg q \vee r)$

$\Longleftrightarrow \neg p \vee \left[(\neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \right]$ By Associative Law

$\Longleftrightarrow \neg p \vee \left[\neg q \vee (p \wedge q) \right] \wedge \neg r$ By distributive law

$\Longleftrightarrow \neg p \vee \left\{ \left[(\neg q \vee p) \wedge \overset{T_0}{\cancel{(\neg q \vee q)}} \right] \wedge \neg r \right\}$ "

$\Longleftrightarrow \neg p \vee \left\{ (\neg q \vee p) \wedge \neg r \right\}$ Identity law.

$\Longleftrightarrow \left[\neg p \vee (\neg q \vee p) \right] \wedge \left[(\neg p \vee \neg r) \right]$ Associative law.

$\Longleftrightarrow \left[\overset{T_0}{\cancel{(\neg p \vee p)}} \vee \neg q \right] \wedge \left[(\neg p \vee \neg r) \right]$

$\Longleftrightarrow T_0 \wedge (\neg p \vee \neg r)$ Identity law.

$\Longleftrightarrow \neg p \vee \neg r$

$\Longleftrightarrow \neg (p \wedge r)$ DeMorgan's law.

b) $(p \wedge q) \rightarrow r$ $\qquad p \rightarrow q \Leftrightarrow \neg p \vee q$

$\Leftrightarrow \neg(p \wedge q) \vee r$

Negation is $\neg[(p \wedge q) \rightarrow r] \Leftrightarrow \neg[\neg(p \wedge q) \vee r]$

$\neg(p \wedge q) \rightarrow r \Leftrightarrow \underline{\underline{(p \wedge q) \wedge \neg r}}$

2) Prove that $(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p$ with out constructing truth tables.

Reasons

$(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$ De Morgan's Law

$\Leftrightarrow p \vee (q \wedge \neg q)$ Distributive law

$\Leftrightarrow p \vee F_0$ Inverse law

$\Leftrightarrow p$ Identity Law.

3) With out constructing truth tables prove the following.
H.W

$\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow q \wedge r$

H.W qn Answer page no : 21

c) $p \vee q \vee (\neg p \wedge \neg q \wedge r)$

$\neg [p \vee q \vee (\neg p \wedge \neg q \wedge r)] \Leftrightarrow \neg p \wedge \neg q \wedge \neg (\neg p \wedge \neg q \wedge r)$

$\Leftrightarrow [\neg p \wedge \neg q] \wedge (p \vee q \vee \neg r)$

$\Leftrightarrow \neg (p \vee q) \wedge (p \vee q \vee \neg r)$

$\Leftrightarrow [\neg (p \vee q) \overset{\to F_0}{\wedge} (p \vee q)] \vee [\neg (p \vee q) \wedge \neg r]$

$\Leftrightarrow \neg (p \vee q) \wedge \neg r$

$\Leftrightarrow \neg p \wedge \neg q \wedge \neg r$

$\Leftrightarrow \underline{\neg (p \vee q \vee r)}$

H.W qn Answer page no: 22

3) $\neg [\neg [(p \vee q) \wedge r] \vee \neg q]$

$\Leftrightarrow (p \vee q) \wedge r \wedge q$ De Morgan's Law

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$ Associative Law

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ Commutative Law

$\Leftrightarrow \underline{q \wedge r}$ Absorption Law.

LOGICAL IMPLICATION: RULES OF INFERENCE

Logical Implication: (Defn) ($\Rightarrow$)

A statement p is said to logically imply another statement q if and only if $p \to q$ is a tautology. This is denoted by $p \Rightarrow q$ and read as p logically implies q

ie $p \Rightarrow q$ iff $p \to q$ is a tautology.

Argument (Defn)

An argument is an assertion that a given set of propositions $P_1, P_2, P_3 \cdots P_n$ yields another proposition q.

Such an argument is denoted by

$$P_1, P_2, P_3 \cdots P_n \vdash q$$

$P_1, P_2, P_3 \cdots P_n$ are called premises and the statement q is called the conclusion for the argument.

Validity of an argument

An argument $P_1, P_2, \cdots P_n \vdash q$ is said to be valid iff $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to q$ is a tautology.

ie $P_1, P_2, \cdots P_n \vdash q$ is valid iff $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Rightarrow q$

Fallacy

An argument which is not valid is called a fallacy.

1. Show that $[p \wedge ((p \wedge r) \rightarrow s)] \Rightarrow (r \rightarrow s)$ using T.T

Ans:- Let the premises be $P_1 : p$

$P_2 : (p \wedge r) \rightarrow s$

Let the conclusion be $q : r \rightarrow s$

We have to prove that $(P_1 \wedge P_2) \rightarrow q$ is a T.(Tautology)

P_1				P_2	q	$(P_1 \wedge P_2)$	$(P_1 \wedge P_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$		$[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow q$
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	0	1	1	0	1
0	1	0	0	1	0	0	1
0	0	1	0	1	1	0	1
0	0	0	0	1	1	0	1

Tautology.

It is proved that $(P_1 \wedge P_2) \rightarrow q$ is a tautology (last column)

ie $[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$ is a

$\therefore [p \wedge ((p \wedge r) \rightarrow s)] \Rightarrow (r \rightarrow s)$

Note :-

For the above problem, we can say that the argument $p, (p \wedge r) \rightarrow s \vdash (r \rightarrow s)$ is valid.

ie The conclusion $(r \rightarrow s)$ is derived from the premises $p, (p \wedge r) \rightarrow s$.

Another method to check the validity of an argument

With out checking the Tautology (By constructing premises & conclusion only)

If a set of premises and conclusion are given it is possible to determine whether the conclusion follows from the premises by Truth Table

Eg:-1) $P_1 : \neg P$ $P_2 : P \vee q$ conclusion : q Check the validity?

Ans:-

		P_1	P_2	Conclusion
P	q	$\neg P$	$P \vee q$	q
1	1	0	1	1
1	0	0	1	0
0	1	1	1	1
0	0	1	0	0

Here we could be concerned with the result in row 3. In that row, the truth value of premises P_1 and P_2 are true (No need to consider other rows) consequently conclusion is also true.

Hence $\neg P, P \vee q \vdash q$ is a valid argument

2) $(\neg P) \wedge (P \vee q) \Rightarrow P \wedge q$ Check the logical implication.

Ans:-

		P_1	P_2	C
P	q	$\neg P$	$P \vee q$	$P \wedge q$
1	1	0	1	1
1	0	0	1	0
0	1	1	1	0
0	0	1	0	0

Here P_1 & P_2 are true in 3rd row but conclusion c is not not true. Hence $P \wedge q$ doesn't follow from $\neg P$ and $P \vee q$.

ie $\neg P \wedge (P \vee q) \not\Rightarrow P \wedge q$ (not a valid argument)

A fallacy

Problems

1. The following are 3 valid arguments. Establish the validity of each by means of a truth table. In each case, determine which rows of the table are crucial for assessing the validity of the argument and which rows can be ignored.

a) $[p \wedge ((p \rightarrow q) \wedge r)] \rightarrow [(p \vee q) \rightarrow r]$

H.W b) $[[(p \wedge q) \rightarrow r] \wedge \neg q \wedge (p \rightarrow \neg r)] \rightarrow (\neg p \vee \neg q)$

H.W c) $[[p \vee (q \vee r)] \wedge \neg q] \rightarrow (p \vee r)$

Ans:- Let $P_1 : p$, $P_2 : (p \rightarrow q) \wedge r$ conclusion : $(p \vee q) \rightarrow r$

P_1				P_2		C
p	q	r	$p \rightarrow q$	$(p \rightarrow q) \wedge r$	$(p \vee q)$	$(p \vee q) \rightarrow r$
1	1	1	1	1	1	1
1	1	0	1	0	1	0
1	0	1	0	0	1	1
1	0	0	0	0	1	0
0	1	1	1	1	1	1
0	1	0	1	0	1	0
0	0	1	1	1	0	1
0	0	0	1	0	0	1

The validity of the argument follows from the first row. Last seven rows may be ignored.

H.W answers 30

1) b)

				P_2			P_1	P_3	C	
p	q	r	¬p	¬q	¬r	p∧q	p∧q→r	p→¬r	¬p∨¬q	
1	1	1	0	0	0	1	1	0	0	
1	1	0	0	0	1	1	0	1	0	
1	0	1	0	1	0	0	1	0	1	
1	0	0	0	1	1	0	1	1	1	row 4
0	1	1	1	0	0	0	1	1	1	
0	1	0	1	0	1	0	1	1	1	
0	0	1	1	1	0	0	1	1	1	row 7
0	0	0	1	1	1	0	1	1	1	row 8

The validity of the argument follows from rows <u>4, 7, 8</u>. Remaining rows may be ignored.

1) c)

			P_2	P_1	C	
p	q	r	¬q	p∨(q∨r)	p∨r	
1	1	1	0	1	1	
1	1	0	0	1	1	
1	0	1	1	1	1	→ row 3
1	0	0	1	1	1	→ row 4
0	1	1	0	1	1	
0	1	0	0	1	1	
0	0	1	1	1	1	→ row 7
0	0	0	1	0	0	

The rows <u>3, 4, 7</u> establish the validity of the given argument. The results in the other 5 rows can be ignored.

2) Use truth tables to verify that each of the following is a logical implication.

a) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

H.W b) $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

Ans:-

2 a)

			P_1	P_2	C
p	q	r	$p \to q$	$q \to r$	$p \to r$
1	1	1	1	1	1
1	1	0	1	0	0
1	0	1	0	1	0
1	0	0	0	1	0
0	1	1	1	1	1
0	1	0	1	0	1
0	0	1	1	1	1
0	0	0	1	1	1

The validity of the argument follows from rows 1, 5, 7, 8

In these rows P_1 and P_2 are true, consequently the conclusion also true.

Hence $(p \to q) \wedge (q \to r) \Longrightarrow p \to r$

H.W

2 b)

			P_1	P_2		C
p	q	r	$p \to r$	$q \to r$	$p \vee q$	$p \vee q \to r$
1	1	1	1	1	1	1
1	1	0	0	0	1	0
1	0	1	1	1	1	1
1	0	0	0	1	1	0
0	1	1	1	1	1	1
0	1	0	1	0	1	0
0	0	1	1	1	0	1
0	0	0	1	1	0	1

$(p \to r) \wedge (q \to r) \Longrightarrow (p \vee q) \to r$

follows from rows 1, 3, 5, 7, 8

3. Verify that each of the following is a logical implication by showing that it is impossible for the conclusion to have the truth value 0 while the hypothesis (premises) has truth value 1

H.W

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

c) $[(p \vee q) \wedge \neg p] \rightarrow q$

RULES OF INFERENCE

Rule of Inference	Related Logical Implication	Name of Rule
1) p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Rule of Detachment (Modus Ponens)
2) $p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Law of the Syllogism
3) $p \to q$ $\neg q$ $\therefore \neg p$	$[(p \to q) \wedge \neg q] \to \neg p$	Modus Tollens
4) p q $\therefore p \wedge q$		Rule of Conjunction
5) $p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Rule of Disjunctive Syllogism
6) $\neg p \to F_0$ $\therefore p$	$(\neg p \to F_0) \to p$	Rule of Contradiction
7) $p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Rule of Conjunctive Simplification
8) p $\therefore p \vee q$	$p \to p \vee q$	Rule of Disjunctive Amplification
9) $p \wedge q$ $p \to (q \to r)$ $\therefore r$	$[(p \wedge q) \wedge [p \to (q \to r)]] \to r$	Rule of Conditional Proof
10) $p \to r$ $q \to r$ $\therefore (p \vee q) \to r$	$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$	Rule for Proof by Cases
11) $p \to q$ $r \to s$ $p \vee r$ $\therefore q \vee s$	$[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$	Rule of the Constructive Dilemma
12) $p \to q$ $r \to s$ $\neg q \vee \neg s$ $\therefore \neg p \vee \neg r$	$[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$	Rule of the Destructive Dilemma

Eg:1

1) Our first example demonstrates the validity of the argument

$$\begin{array}{l} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \\ \hline \therefore \neg r \rightarrow s \end{array}$$

Steps	**Reasons**
1) $p \rightarrow r$	Premise
2) $\neg r \rightarrow \neg p$	Step (1) and $p \rightarrow r \Leftrightarrow \neg r \rightarrow \neg p$
3) $\neg p \rightarrow q$	Premise
4) $\neg r \rightarrow q$	Steps (2) and (3) and the Law of the Syllogism
5) $q \rightarrow s$	Premise
6) $\therefore \neg r \rightarrow s$	Steps (4) and (5) and the Law of the Syllogism

A second way to validate the given argument proceeds as follows.

Steps	**Reasons**
1) $p \rightarrow r$	Premise
2) $q \rightarrow s$	Premise
3) $\neg p \rightarrow q$	Premise
4) $p \vee q$	Step (3) and $(\neg p \rightarrow q) \Leftrightarrow (\neg\neg p \vee q) \Leftrightarrow (p \vee q)$, where the second logical equivalence follows by the Law of Double Negation
5) $r \vee s$	Steps (1), (2), and (4) and the Rule of the Constructive Dilemma
6) $\therefore \neg r \rightarrow s$	Step (5) and $(r \vee s) \Leftrightarrow (\neg\neg r \vee s) \Leftrightarrow (\neg r \rightarrow s)$, where the Law of Double Negation is used in the first logical equivalence

Eg:2

2) Establish the validity of the argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

Steps	Reasons
1) $p \to q$	Premise
2) $q \to (r \wedge s)$	Premise
3) $p \to (r \wedge s)$	Steps (1) and (2) and the Law of the Syllogism
4) $p \wedge t$	Premise
5) p	Step (4) and the Rule of Conjunctive Simplification
6) $r \wedge s$	Steps (5) and (3) and the Rule of Detachment
7) r	Step (6) and the Rule of Conjunctive Simplification
8) $\neg r \vee (\neg t \vee u)$	Premise
9) $\neg(r \wedge t) \vee u$	Step (8), the Associative Law of $\vee$, and DeMorgan's Laws
10) t	Step (4) and the Rule of Conjunctive Simplification
11) $r \wedge t$	Steps (7) and (10) and the Rule of Conjunction
12) $\therefore u$	Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism

3) Eg:3

This example will provide a way to show that the following argument is valid.

If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry. If the party were canceled, then refunds would have had to be made. No refunds were made.

Therefore the band could play rock music.

First we convert the given argument into symbolic form by using the following statement assignments:

p: The band could play rock music.
q: The refreshments were delivered on time.
r: The New Year's party was canceled.
s: Alicia was angry.
t: Refunds had to be made.

The argument above now becomes

$$(\neg p \vee \neg q) \to (r \wedge s)$$
$$r \to t$$
$$\neg t$$
$$\overline{\therefore p}$$

Steps	Reasons
1) $r \to t$	Premise
2) $\neg t$	Premise
3) $\neg r$	Steps (1) and (2) and Modus Tollens
4) $\neg r \vee \neg s$	Step (3) and the Rule of Disjunctive Amplification
5) $\neg(r \wedge s)$	Step (4) and DeMorgan's Laws
6) $(\neg p \vee \neg q) \to (r \wedge s)$	Premise
7) $\neg(\neg p \vee \neg q)$	Steps (6) and (5) and Modus Tollens
8) $p \wedge q$	Step (7), DeMorgan's Laws, and the Law of Double Negation
9) $\therefore p$	Step (8) and the Rule of Conjunctive Simplification

4) Eg:4

Show that

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \to r \\ \neg r \\ \hline \therefore p \end{array}$$

Steps	Reasons
1) $\neg p \leftrightarrow q$	Premise
2) $(\neg p \to q) \wedge (q \to \neg p)$	Step (1) and $(\neg p \leftrightarrow q) \Leftrightarrow [(\neg p \to q) \wedge (q \to \neg p)]$
3) $\neg p \to q$	Step (2) and the Rule of Conjunctive Simplification
4) $q \to r$	Premise
5) $\neg p \to r$	Steps (3) and (4) and the Law of the Syllogism
6) $\neg p$	Premise (the one assumed)
7) r	Steps (5) and (6) and the Rule of Detachment
8) $\neg r$	Premise
9) $r \wedge \neg r (\Leftrightarrow F_0)$	Steps (7) and (8) and the Rule of Conjunction
10) $\therefore p$	Steps (6) and (9) and the method of Proof by Contradiction

The method of Proof by Contradiction (Indirect method of proof)

Suppose that we are given a set of premises $P_1, P_2, \dots P_n$ and a conclusion C. Start with $\neg C$ (ie we introduce $\neg C$ as an additional premise) Show that $\neg C$ with the set of premises $P_1, P_2, \dots P_n$ leads a contradiction F_0.

Then we say that C is a valid conclusion by the proof of contradiction.

4. For each of the following pairs of statements, use Modus Ponens or Modus Tollens to fill in the blank line so that a valid argument is presented.

a) If Janice has trouble starting her car, then her daughter Angela will check Janice's spark plugs.
Janice had trouble starting her car.
∴ ______________________________

b) If Brady solved the first problem correctly, then the answer he obtained is 137.
Brady's answer to the first problem is not 137.
∴ ______________________________

c) If this is a **repeat-until** loop, then the body of this loop is executed at least once.
____
∴ The body of the loop is executed at least once.

d) If Tim plays basketball in the afternoon, then he will not watch television in the evening.
____
∴ Tim didn't play basketball in the afternoon.

8. Give the reasons for the steps verifying the following argument.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

Steps **Reasons**

1) $\neg s \wedge \neg u$
2) $\neg u$
3) $\neg u \rightarrow \neg t$
4) $\neg t$
5) $\neg s$
6) $\neg s \wedge \neg t$
7) $r \rightarrow (s \vee t)$
8) $\neg(s \vee t) \rightarrow \neg r$
9) $(\neg s \wedge \neg t) \rightarrow \neg r$
10) $\neg r$
11) $(\neg p \vee q) \rightarrow r$
12) $\neg r \rightarrow \neg(\neg p \vee q)$
13) $\neg r \rightarrow (p \wedge \neg q)$
14) $p \wedge \neg q$
15) $\therefore p$

9. **a)** Give the reasons for the steps given to validate the argument

$$[(p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)] \rightarrow (\neg q \rightarrow s).$$

Steps **Reasons**

1) $\neg(\neg q \rightarrow s)$
2) $\neg q \wedge \neg s$
3) $\neg s$
4) $\neg r \vee s$
5) $\neg r$
6) $p \rightarrow q$
7) $\neg q$
8) $\neg p$
9) $p \vee r$
10) r
11) $\neg r \wedge r$
12) $\therefore \neg q \rightarrow s$

Conditional Proof (When conclusion is of the form $q \to r$)

Consider the argument

$$\begin{array}{|c|} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline \therefore q \to r \end{array}$$

Here we can consider another argument as follows

$$\begin{array}{|c|} P_1 \\ P_2 \\ \vdots \\ P_n \\ q \\ \hline \therefore r \end{array}$$

$P_1, P_2, \ldots, P_n$ } premises

$q \to$ Additional premise

Note:- If the conclusion is of the form $q \to r$ we will take q as an additional premise and derive r using the given premises and q.

H.W 1) Derive $p \to (q \to s)$ using conditional proof from the premises $p \to (q \to r)$ and $q \to (r \to s)$

2) $[P \wedge P \rightarrow [q \rightarrow (r \wedge s)]] \longrightarrow (q \rightarrow s)$ using conditional proof

Steps	Reasons
① P	Premise
② $P \rightarrow [q \rightarrow (r \wedge s)]$	Premise
③ q	Additional premise (Conditional proof)
④ $q \rightarrow (r \wedge s)$	①, ② Modus Ponens
⑤ $r \wedge s$	③, ④ " "
⑥ S	⑤ Rule of ~~conjunction~~ simplification.

$\therefore$ We can say that $P \wedge P \rightarrow [(q \rightarrow (r \wedge s)] \Rightarrow (q \rightarrow s)$

or $q \rightarrow s$ is a valid conclusion.

More problems.

3) Establish the validity of the argument

H.W

$[(P \wedge \neg q) \wedge r] \longrightarrow [(P \wedge r) \vee q]$

4) $[P \wedge (P \rightarrow q) \wedge (\neg q \vee r)] \longrightarrow r$

5)
$$\begin{array}{l} P \vee q \\ \neg P \vee r \\ \hline \therefore q \end{array}$$

6)
$$\begin{array}{l} P \rightarrow q \\ r \rightarrow \neg q \\ r \\ \hline \therefore \neg P \end{array}$$

P. 39 Answer

1) $p \to (q \to r)$

$q \to (r \to s)$

Conclusion $\therefore \boxed{p \to (q \to s)}$

using CP Rule.

Consider p as an additional premise & derive $q \to s$

Steps	Reasons.
① p	Additional Premise
② $p \to (q \to r)$	premise
③ $q \to r$	①,② Modus Ponens
④ $q \to (r \to s)$	Premise
⑤ $\neg q \vee (r \to s)$	④, $p \to (r \to s) \Leftrightarrow \neg p \vee (r \to s)$
⑥ $\neg q \vee (\neg r \vee s)$	⑤ $\neg r \vee s \Leftrightarrow r \to s$
⑦ $\neg r \vee (\neg q \vee s)$	⑥ Commutative & Associative laws
⑧ $r \to (\neg q \vee s)$	⑦ $\neg r \vee (\neg q \vee s) \Leftrightarrow r \to (\neg q \vee s)$
⑨ $r \to (q \to s)$	⑧ $\neg q \vee s \Leftrightarrow q \to s$
⑩ $q \to (q \to s)$	③,⑨, Law of syllogism
⑪ $\neg q \vee (\neg q \vee s)$	⑩ $q \to (q \to s) \Leftrightarrow \neg q \vee (\neg q \vee s)$
⑫ $(\neg q \vee \neg q) \vee s$	⑪ Associative
⑬ $\neg q \vee s$	⑫ idempotent law $p \vee p \Leftrightarrow p$
⑭ $q \to s$.	⑬ $\neg q \vee s \Leftrightarrow q \to s$
⑮ $\therefore p \to (q \to s)$	⑭ ①, Rule CP

After page 38 (For each of the following pairs of statements use Modus Ponens or Tollens to fill in the blank lines . . .)

Answers

4. ⓐ p: Janice has trouble starting her car
q: Angella will check Janice's spark plugs
symbolic form of the argument is

$p \to q$
p
$\therefore q$ Angela will check Janices spark plugs.

$\boxed{\text{Modus Ponens}}$

ⓑ p: Brady solved the first problem correctly
q: The answer Brady obtained is 137

Argument is
$p \to q$
$\neg q$
$\therefore \neg p$: Brady didn't solve the first prob

$\boxed{\text{Modus Tollens}}$

ⓒ $p \to q$
p : This is a "repeat-until" loop
$\therefore q$

$\boxed{\text{Modus Ponens}}$

ⓓ $p \to q$
$\neg q$: Tim will watch T.V in the evening
$\neg p$

$\boxed{\text{Modus Tollens}}$

Q. Give ~~reasons~~ direct proof for argument ~~...~~ in page ~~34~~ 37

$\neg p \leftrightarrow q$
$q \to r$
$\neg r$
$\therefore p$

Ans

Steps	Reasons
① $\neg p \leftrightarrow q$	premise
② $(\neg p \to q) \wedge (q \to \neg p)$	①, $\neg p \leftrightarrow q \Leftrightarrow (\neg p \to q) \wedge (q \to \neg p)$ law of equivalence
③ $q \to r$	premise
④ $\neg r$	premise
⑤ $\neg p \to q$	② Simplification
⑥ $\neg p \to r$	⑤, ③ Law of syllogism
⑦ $\neg r \to p$	⑥ Contrapositive of $\neg p \to r \Longleftrightarrow \neg r \to p$
⑧ $\therefore p$	⑦, ④ Modus ponens

9. a)

1)	Premise (The Negation of the Conclusion)
2)	Step (1) and $\neg(\neg q \rightarrow s) \Leftrightarrow \neg(\neg\neg q \vee s) \Leftrightarrow \neg(q \vee s) \Leftrightarrow \neg q \wedge \neg s$
3)	Step (2) and the Rule of Conjunctive Simplification
4)	Premise
5)	Steps (3) and (4) and the Rule of Disjunctive Syllogism
6)	Premise
7)	Step (2) and the Rule of Conjunctive Simplification
8)	Steps (6) and (7) and Modus Tollens
9)	Premise
10)	Steps (8) and (9) and the Rule of Disjunctive Syllogism
11)	Steps (5) and (10) and the Rule of Conjunction
12)	Step (11) and the Method of Proof by Contradiction

8. Give the reasons for the steps verifying the following argument.

$$(\neg p \vee q) \rightarrow r$$
$$r \rightarrow (s \vee t)$$
$$\neg s \wedge \neg u$$
$$\underline{\neg u \rightarrow \neg t}$$
$$\therefore p$$

Steps	**Reasons**
1) $\neg s \wedge \neg u$	premise
2) $\neg u$	①, Simplification
3) $\neg u \rightarrow \neg t$	premise
4) $\neg t$	②③ Modus ponens
5) $\neg s$	① Simplification
6) $\neg s \wedge \neg t$	④⑤ conjunction
7) $r \rightarrow (s \vee t)$	premise
8) $\neg(s \vee t) \rightarrow \neg r$	⑦ $r \rightarrow (s \vee t) \Leftrightarrow \neg(s \vee t) \rightarrow \neg r$
9) $(\neg s \wedge \neg t) \rightarrow \neg r$	⑧ DeMorgans law
10) $\neg r$	⑥,⑨ Modus ponens
11) $(\neg p \vee q) \rightarrow r$	premise
12) $\neg r \rightarrow \neg(\neg p \vee q)$	⑪ contrapositive
13) $\neg r \rightarrow (p \wedge \neg q)$	⑫ DeMorgan's law
14) $p \wedge \neg q$	⑩⑬ Modus Ponens
15) $\therefore p$	⑭ Simplification

3) Establish the validity of the following argument

$[(p\wedge\neg q)\wedge r] \to (p\wedge r)\vee q$

Steps	Reasons
① $p\wedge\neg q$	premise
② p	① simplification
③ $\neg q$	① simplification.
④ r	premise
⑤ $p\wedge r$	②,④ conjunction.
⑥ $\therefore (p\wedge r)\vee q$	⑤ Amplification $p\wedge r \to (p\wedge r)\vee q$

4) $[p\wedge(p\to q)\wedge(\neg q\vee r)] \to r$

Steps	Reasons
① p	premise
② $p\to q$	premise
③ q	①,② Modus ponens.
④ $\neg q\vee r$	premise
⑤ $q\to r$	④, $\neg q\vee r \Leftrightarrow q\to r$
⑥ $\therefore r$	③,⑤ Modus Ponens.

5) $p\vee q$

$\neg p\vee r$

$\neg r$ (Correction)

$\therefore q$

Steps	Reasons
① $p\vee q$	premise
② $\neg p\vee r$	premise
③ $p\to r$	② $\neg p\vee r \Leftrightarrow p\to r$
④ $\neg r$	premise
⑤ $\neg p$	③,④ Modus Tollens.
⑥ $\neg p\to q$	① $p\vee q \Leftrightarrow \neg p\to q$
⑦ $\therefore q$	⑤,⑥ Modus Ponens.

6) $p\to q$

$r\to\neg q$

r

$\therefore \neg p$

Steps	Reasons
① $p\to q$	premise
② $r\to\neg q$	premise
③ r	premise
④ $\neg q$	②,③ Modus Ponens
⑤ $\therefore \neg p$	④,① Modus Tollens

7. Show that each of the following arguments is invalid by providing a counter example. ie, an assignment of the truth value for the given primitive statements p, q, r, s such that all premises are true (have the truth value 1) while the conclusion is false (has the truth value 0)

(a) $[(p \wedge \neg q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow \neg r$

H.W (b) $[(p \wedge q) \rightarrow r] \wedge [(\neg q \vee r)] \rightarrow p$

H.W (c)
p
$p \rightarrow r$
$p \rightarrow (q \vee \neg r)$
$\neg q \vee \neg s$
$\therefore s$

Ans a)

P				P_1		P_2	C
p	q	r	$\neg q$	$p \wedge \neg q$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$\neg r$
1	1	1	0	0	1	1	0
1	1	0	0	0	0	0	1
1	0	1	1	1	1	1	0
1	0	0	1	1	1	1	1
0	1	1	0	0	1	1	0
0	1	0	0	0	0	1	1
0	0	1	1	0	1	1	0
0	0	0	1	0	1	1	1

From the 3rd row the argument is not valid.
ie $p:1,\ q:0,\ r:1$ is the counter example.

12. Write each of the following arguments in symbolic form. Then establish the validity of the argument or give a counter-example to show that it is invalid.

a) If Rochelle gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

b) If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to the racetrack and Ralph didn't play cards all night.

c) If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 80°F, there is no chance for rain. Today the temperature is 85°F and Lois is wearing her red headband. Therefore (sometime today) Lois will mow her lawn.

13. **a)** Given primitive statements p, q, r, show that the implication

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

is a tautology.

RESULT (Resolution)

$p \vee q$
$\neg p \vee r$ } premises are called clauses

$\therefore q \vee r$ } The conclusion is called resolvent.

8) Write the following argument in symbolic form and establish its validity using resolution.
Jonathan doesnot have his driver's license or his new car is out of gas. Jonathan has his driver's license or he doesn't like to drive his new car. Jonathan's new car is not out of gas or he doesnot like to drive his new car. Therefore, Jonathan doesnot like to drive his new car.

Ans:- p: Jonathan has his driver's license
q: Jonathan's new car is out of gas.
r: Jonathan likes to drive his car.

The given argument can be written in symbolic form as

$$\neg p \vee q$$
$$p \vee \neg r$$
$$\neg q \vee \neg r$$
$$\overline{\therefore \neg r}$$

Quantifiers

A declarative sentence is an *open statement* if

1) it contains one or more variables, and

2) it is not a statement, but

3) it becomes a statement when the variables in it are replaced by certain allowable choices.(universe of discourse).

Eg:

The open statement "The number $x+2$ is an even integer" is denoted by $p(x)$ [or $q(x)$, $r(x)$, etc.]. Then $\neg p(x)$ may be read "The number $x+2$ is *not* an even integer."

$q(x, y)$: The numbers $y+2$, $x-y$, and $x+2y$ are even integers.

$p(5)$: The number $7(=5+2)$ is an even integer. (FALSE)

$\neg p(7)$: The number 9 is not an even integer. (TRUE)

$q(4, 2)$: The numbers 4, 2, and 8 are even integers. (TRUE)

Consequently, we see that for both $p(x)$ and $q(x, y)$, as already given, some substitutions result in true statements and others in false statements. Therefore we can make the following true statements.

1) For some x, $p(x)$.

2) For some x, y, $q(x, y)$.

The phrases "For some x" and "For some x, y" are said to *quantify* the open statements $p(x)$ and $q(x, y)$, respectively.

There are two types of Quantifiers

1. Existential Quantifier ($\exists$)

The phrases are "for some x", "for some x, y" "for atleast one x", "There exists an x such that"

Eg:- For some x, $p(x)$ becomes $\exists x\, p(x)$ in symbolic form.

Eg:- For some x, y, $q(x,y)$ becomes $\exists x \exists y\, q(x,y)$ in symbolic form

This one can be abbreviate as $\exists x, y\, q(x,y)$

2. The Universal quantifier ($\forall$)

The phrases used for universal quantifier are

For all x

For any x

For each x

For every x, y etc.

Eg:- 1) For all x, $p(x)$ becomes $\forall x\ p(x)$ in symbolic form

2) For every x, y $q(x,y)$ becomes $\forall x \forall y\ q(x,y)$.

It can be abbreviated to $\forall xy\ q(x,y)$

3) Let $r(x)$: $2x$ is an even integer (with universe of all integers)

quantified statement is

$\forall x\ r(x)$ and true also.

The universe or universe of discourse

The allowable choices or values for variables in an open statement is called the universe or universe of discourse. Actually it is a set.

Eg:- $p(x): x \geq 0$, The universe comprise all real numbers.

The set under consideration or, x can take all real numbers.

1) Let $p(x), q(x)$ denote the following open statements.
$p(x): x \leq 3$ $\quad q(x): x+1$ is odd. If the universe consists of all integers. What are the truth values of the following statements.

a) $q(1)$ b) $\neg p(3)$ c) $p(7) \vee q(7)$
d) $p(3) \wedge q(4)$ e) $\neg [p(-4) \vee q(-3)]$ f) $\neg p(-4) \wedge \neg q(-3)$

Ans a) F b) F c) F d) T e) F f) F

2) Determine all values of x for which $[p(x) \wedge q(x)] \wedge r(x)$ results in a true statement

where $p(x): x \leq 3$
$q(x): x+1$ is odd
$r(x): x > 0$

$p(x): x \leq 3$ is true for $\{1, 2, 3\}$
$q(x): x+1$ is odd true for all even integers. $\{2, 4, 6, \cdots 8\}$
$r(x): x > 0$ is true for $\{1, 2, 3, \cdots\}$

$\therefore$ Value of x where the statement is true $\{2\}$

3) Determine the truth values of the following statements, where $p(x)$, $q(x)$, $r(x)$ are same in Q.(2).

a) $p(3) \vee [q(3) \vee \neg r(3)]$ $T \vee (F \vee F)$ ∴ True

b) $p(2) \longrightarrow [q(2) \to r(2)]$ $T \to (T \to T)$ ∴ True.

4) Let $p(x)$ be $x^2 = 2x$ where the universe comprises all integers. Check whether the following are true or false.

a) $p(0)$
b) $p(1)$
c) $p(2)$
d) $p(-2)$
e) $\exists x \, p(x)$
f) $\forall x \, p(x)$

Ans:-
a) T
b) F
c) T
d) F
e) T
f) F

Eg: 1) Consider $[\sin^2 x + \cos^2 x = 1]$ the trigonometric identity.

The identity can be expressed by the quantified statement $\forall x [\sin^2 x + \cos^2 x = 1]$ [universal quantifier.

This quantified statement is true for all real numbers x (even if the universe is specified)

2) Consider the universe of all +ve integers and the sentence

" The integer 41 is equal to the sum of two perfect squares"

The quantified statement is like this

$\exists m \exists n [41 = m^2 + n^2]$ [existential quantifier].

Statement	When it is true
$\exists x\, P(x)$	For some (at least one) x in the universe, $P(x)$ is true
$\forall x\, P(x)$	For every x from the universe $P(x)$ is true
$\exists x \neg P(x)$	For at least one x in the universe, $P(x)$ is false so $\neg P(x)$ is true
$\forall x \neg P(x)$	For all x from the universe $P(x)$ is false or $\neg P(x)$ is true.

SUMMARY

Definition Contra positive, Converse & Inverse of $\forall x\,[p(x) \rightarrow q(x)]$

1) $\forall x\,[\neg q(x) \rightarrow \neg p(x)]$ Contrapositive

2) $\forall x\,[q(x) \rightarrow p(x)]$ Converse

3) $\forall x\,[\neg p(x) \rightarrow \neg q(x)]$ Inverse.

LOGICAL EQUIVALENCES & LOGICAL IMPLICATIONS

1) $\exists x\,[p(x) \wedge q(x)] \xRightarrow{\text{logically implies}} [\exists x\, p(x) \wedge \exists x\, q(x)]$

2) $\exists x\,[p(x) \vee q(x)] \xLeftrightarrow{\text{logically equivalent}} [\exists x\, p(x) \vee \exists x\, q(x)]$

3) $\forall x\,[p(x) \wedge q(x)] \xLeftrightarrow{\text{logically equivalent}} [\forall x\, p(x) \wedge \forall x\, q(x)]$

4) $[\forall x\, p(x) \vee \forall x\, q(x)] \xRightarrow{\text{logically implies}} \forall x\,[p(x) \vee q(x)]$

Negation of the quantifiers

$\neg [\forall x\, p(x)] \Leftrightarrow \exists x\, \neg p(x)$

$\neg [\exists x\, p(x)] \Leftrightarrow \forall x\, \neg p(x)$

$\neg [\forall x\, \neg p(x)] \Leftrightarrow \exists x\, p(x)$

$\neg [\exists x\, \neg p(x)] \Leftrightarrow \forall x\, p(x)$

Note :-

1. While representing statement using universal quantifiers ($\forall$) we commonly use the connective "if... then" ($\longrightarrow$)

2. While representing statement using existential quantifier ($\exists$) we commonly use the connective "and" "(but)" ($\wedge$)

Quantifier	Connective
$\forall$ For all, every	$\longrightarrow$ if .... then
$\exists$ There exists, atleast	$\wedge$ and, (but)

More Problems.

1) For the universe of all integers, let $p(x), q(x), r(x), s(x)$ and $t(x)$ be the following open statements

$p(x): x > 0$
$q(x): x$ is even
$r(x): x$ is a perfect square
$s(x): x$ is (exactly) divisible by 4
$t(x): x$ is (exactly) divisible by 5.

(a) Write the following statements in symbolic form and determine the truth value.

(i) At least one integer is even.

Ans:- There exists an integer x such that x is even.
symbolic form is $\exists x\ p(x)$.

(ii) There exists a +ve integer that is even.

Ans:- There exist an integer x such that x is +ve and is even.
symbolic form is $\exists x\ [p(x) \wedge q(x)]$

(iii) If x is even, then x is not divisible by 5.

Ans:- symbolic form is $\forall x\ [q(x) \longrightarrow \neg t(x)]$

(iv) No even number is divisible by 5.

Ans:- The above statement can be rewritten as
For all x, if x is even then it is not divisible by 5.
$\therefore$ symbolic form is $\forall x\ [q(x) \longrightarrow \neg t(x)]$

(v) There exists an even integer divisible by 5.

Ans:- The above statement can be written as,
"There exists an x such that x is even and it is divisible by 5."
Symbolic form is $\exists x\ [q(x) \wedge t(x)]$

(vi) If x is even and x is a perfect square, then x is divisible by 4.

$$\forall x\ [(q(x) \wedge r(x)) \longrightarrow s(x)]$$

(b) Express each of the following symbolic representations in words.

(i) $\forall x\ [r(x) \rightarrow p(x)]$

H.W (ii) $\forall x\ [s(x) \rightarrow q(x)]$

H.W (iii) $\forall x\ [s(x) \rightarrow \neg t(x)]$

(iv) $\exists x\ [s(x) \wedge \neg r(x)]$

Ans:- (i) If x is a perfect square then $x > 0$

(ii) If x is

(iii)

(iv) There exists an integer that is divisible by 4, but it is not a perfect square.

Note:- The phrase 'but' can be used in the place of 'and'.

2. Let $p(x): x^2 - 7x + 10 = 0$ $\quad (x-5)(x-2) = 0 \quad x = 5, 2$

$q(x): x^2 - 2x - 3 = 0$ $\quad (x-3)(x+1) = 0 \quad x = 3, -1$

$r(x): x < 0$

Determine the truth value of the following statements, where the universe is all integers. If a statement is false, provide a counterexample or explanation.

(i) $\forall x\, [p(x) \rightarrow \neg r(x)]$

(ii) $\forall x\, [q(x) \rightarrow r(x)]$

(iii) $\exists x\, [q(x) \rightarrow r(x)]$

(iv) $\exists x\, [p(x) \rightarrow r(x)]$

Ans:- (i) True (If $x = 5$ or 2, then $x \geq 0$ is the statement)

(ii) False (If $x = 3$ or -1, then $x < 0$ is the statement)
ie when $x = 3$, $x < 0$ is false.

(iii) True (There exists an integer $x = 3$ or -1 such that $x < 0$)

(iv) False (False because when $x = 5$, $x = 2$, $x < 0$ is not possible.)

H.W

3. Let $p(x)$, $q(x)$ and $r(x)$ denote the following open statements

$p(x): x^2-8x+15=0$ $\quad (x-3)(x-5)=0 \quad x=5,3$

$q(x): x$ is odd

$r(x): x>0$

For the universe of all integers, determine the truth value of each of the following statements. If the statement is false, give a counterexample

a) $\forall x\,[p(x) \rightarrow q(x)]$

b) $\forall x\,[q(x) \rightarrow p(x)]$

c) $\exists x\,[p(x) \rightarrow q(x)]$

d) $\exists x\,[q(x) \rightarrow p(x)]$

e) $\exists x\,[r(x) \rightarrow p(x)]$

f) $\forall x\,[\neg q(x) \rightarrow \neg p(x)]$

5, 3

g) $\exists x\,[p(x) \rightarrow (q(x) \wedge r(x))]$

h) $\forall x\,[(p(x) \vee q(x)) \rightarrow r(x)]$

4) Identify the bound variables & free variables in each of the following. In both cases the universe comprises all real numbers.

(a) $\exists z \ \forall y \ [\cos(x+y) = \sin(z-x)]$

(b) $\exists x \ \exists y \ (x^2 - y^2 = z)$

Ans: (a) x is a free variable
y, z are bound variables.

(b) z is free
x, y are bound.

<u>Note</u>:- The variables which are quantified ($\exists, \forall$ symbols) are called bound variables. otherwise they are free.

5. Let the universe be set of all real nos. In each case Negate & simplify the given statement

a) $\forall x \ \forall y \ [(x > y) \longrightarrow (x - y > 0)]$

b) $\forall x \ \forall y \ [(x < y) \longrightarrow \exists z \ (x < z < y)]$

c) $\forall x \ \forall y \ [(|x| = |y|) \longrightarrow (y = \pm x)]$
H.W

a) $p(x,y): x > y$

$q(x,y): x - y > 0$

$\forall x \; \forall y \; (p \rightarrow q)$

Its negation is $\neg \left[\forall x \; \forall y \; (\neg p \vee q)\right]$

$\Longleftrightarrow \exists x \; \exists y \; \neg (\neg p \vee q)$

$\Longleftrightarrow \exists x \; \exists y \; (p \wedge \neg q)$

ie $\exists x \; \exists y \left[(x > y) \wedge (x - y \leq 0)\right]$

b) $p(x,y): x < y$

$q(x,y,z): x < z < y$

$\forall x \; \forall y \left[x < y \rightarrow \exists z \; (x < z < y)\right]$

$\forall x \; \forall y \left[p \rightarrow \exists z \; q\right]$

Its negation is $\neg \; \forall x \; \forall y \left[p \rightarrow \exists z \; q\right]$

$\Longleftrightarrow \exists x \; \exists y \; \neg (\neg p \vee \exists z \; q)$

$\Longleftrightarrow \exists x \; \exists y \; (p \wedge \forall z \; \neg q)$

$\Longleftrightarrow \exists x \; \exists y \left[(x < y) \wedge \forall z \; (x \geq z \geq y)\right]$

6. Determine the truth value of each of the following statement if the universe comprises of all real numbers.

a) $\forall x\,[x^2 > x]$

b) $\exists x\,[\text{If } x>1 \text{ then } x^2 > x]$

c) $\forall x\,\forall y\,[x^2 < y+1]$

d) $\exists x\,\exists y\,[x^2 < y+1]$

e) $\forall x\,\forall y\,[x^2+y^2 = 9]$

f) $\exists x\,\exists y\,[x^2+y^2 = 9]$

g) $\exists x\,\forall y\,[x^2+y^2 \geq 0]$

h $\forall x\,\exists y\,[\text{If } x<y \text{ then } x^2 < y^2]$

19. For each of the following statements state the converse, inverse, and contrapositive. Also determine the truth value for each given statement, as well as the truth values for its converse, inverse, and contrapositive. (Here "divides" means "exactly divides.")

a) [The universe comprises all positive integers.]
If $m > n$, then $m^2 > n^2$.

b) [The universe comprises all integers.]
If $a > b$, then $a^2 > b^2$.

c) [The universe comprises all integers.]
If m divides n and n divides p, then m divides p.

d) [The universe consists of all real numbers.]
$\forall x\ [(x > 3) \rightarrow (x^2 > 9)]$

e) [The universe consists of all real numbers.]
For all real numbers x, if $x^2 + 4x - 21 > 0$, then $x > 3$ or $x < -7$.

Assignment 1

1. Show the following implication without constructing truth tables.

$$P \wedge Q \Rightarrow P \rightarrow Q$$

2. Write the negation of the following statement.
" If I drive, then I will not walk."

3. Show that $S \vee R$ is tautologically implied by
$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$

4. Discuss indirect method of proof. Show that the following premises are inconsistant.
   (i) If Jack misses many classes through illness, then he fails highschool.
   (ii) If Jack fails high school, then he is uneducated.
   (iii) If Jack reads a lot of books, then he is not uneducated.
   (iv) Jack misses many classes through illness and reads a lot of books.

(Hint: A set of premises $P_1, P_2, \ldots P_n$ is said to be inconsistent if their conjunction logically implies a contradiction. ie $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow F$.)

5) Write the negation of each of the following statements as an English sentence without symbolic notation. (Here the universe consists of all the students at the university where Professor Lenhart teaches.

(a) Every student in Professor Lenhart's C++ class is majoring in computer science or Maths

(b) At least one student in Professor Lenhart's C++ class is a history major.

6) Rewrite each of the following statements in the "if... then" form. Then write the converse, inverse and contrapositive of them, and give the truth values.

(a) Divisibility by 21 is a sufficient condition for divisibility by 7 (universe- all +ve integers)

(b) For every complex number z, z being real is necessary for z^2 to be real.